Extremal dynamics on complex networks: Analytic solutions

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The Bak-Sneppen model displaying punctuated equilibria in biological evolution is studied on random complex networks. By using the rate equation and the random walk approaches, we obtain the analytic solution of the fitness threshold x_c to be $1/(\langle k \rangle_f + 1)$, where $\langle k \rangle_f = \langle k^2 \rangle / \langle k \rangle$ $(=\langle k \rangle)$ in the quenched (annealed) updating case, where $\langle k^n \rangle$ is the nth moment of the degree distribution. Thus, the threshold is zero (finite) for the degree exponent $\gamma < 3$ $(\gamma > 3)$ for the quenched case in the thermodynamic limit. The theoretical value x_c fits well to the numerical simulation data in the annealed case only. Avalanche size, defined as the duration of successive mutations below the threshold, exhibits a critical behavior as its distribution follows a power law, $P_a(s) \sim s^{-3/2}$.

I. INTRODUCTION

Punctuated equilibrium is an evolution taking place through intermittent bursts of activity separating relatively long periods of quiescence, which can be often found in ecological systems [1,2]. Bak and Sneppen (BS) [3] introduced a simple model to mimic such an evolution. The basis of the BS model is to focus on a minimal set of variables that capture the basic features of punctuated equilibrium while ignoring all other details. In the original model, N species are arranged on a one-dimensional chain with periodic boundary conditions. A fitness value x_i is assigned to each site i (species) on the chain, which is a random variable selected in the interval [0,1]. Evolution in the ecological systems is modeled as follows: At each time step, the ecological system is updated by locating the site with the lowest fitness and mutating it by assigning new random numbers to that site and the $K-1$ nearest neighboring sites. Subsequent updating of the lowest fitness value generates spatial and temporal correlations and displays punctuated equilibria. A distinct feature arising through these dynamics can be found in the distribution of fitness values. After a transient period, the distribution of the fitness values has a discontinuity at a threshold $x_c \approx 0.67$; its elements are zero up to x_c and almost the same constant beyond x_c. The threshold x_c is self-organized.

A mean field version of the BS model was introduced [4], in which are updated the minimum fitness value as well as the fitness values of other $K-1$ sites selected at random in the system. Such a modified model enables one to solve the problem analytically. The threshold was obtained to be $x_c = 1/K$ in the limit $N \to \infty$. Also the notion of avalanche was introduced to quantify the correlation between bursts of evolutionary activity. Avalanche size is the time interval between two successive occasions where no fitness value is less than a given value. It was proposed based on the branching process analysis [5] and later derived by using the random walk approach [6,7] that the avalanche size distribution follows a power law as $P_a(s) \sim s^{-3/2}$ when the given value is chosen as the threshold x_c.

Ecological systems in the real world are complex. Interactions between individual species are not as simple as one dimensional, but form a complex network. Thus, it would be interesting to extend previous studies of the BS model performed in the Euclidean space to complex networks such as scale free (SF) networks, while it is still controversial whether the ecological systems such as food webs are SF-networked systems [8]. SF networks mean that the number of connections to each species, called degree k in graph theory, follows a power law, $P_d(k) \sim k^{-\gamma}$ [9–12]. While such an extension is natural, only a few studies in that direction have been performed so far. Christensen *et al.* [13] have studied the BS model on random networks [14]. Kulkarni *et al.* [15] studied it on the small-world network introduced by Watts and Strogatz [16]. Moreno and Vazquez [17] studied the BS model on SF networks with $\gamma = 3$, obtaining that the threshold x_c is given as $x_c = \langle k \rangle / \langle k^2 \rangle$ by using the heuristic argument similar to the one used in the contact process, where $\langle \cdots \rangle$ denotes the average over the degree distribution. They found that x_c depends on the system size N as $x_c \sim 1/\ln N$, so that it vanishes in the limit $N \to \infty$. The N-dependent behavior was obtained numerically at $\gamma = 3$, so that the result may be rooted from logarithmic correction. Recently, Lee and Kim [18] also studied the same problem on SF networks but with general $\gamma > 2$. They obtained that the threshold is given as $x_c = (\langle k \rangle + 1)/\langle (k+1)^2 \rangle$ by using heuristic arguments. Thus the threshold vanishes for $\gamma < 3$ and finite for $\gamma > 3$. The interesting feature they obtained is the crossover behavior in the avalanche size distribution between two different power-law behaviors.

Here we study the BS model on random SF networks analytically by using both the rate equation and the random walk approaches [19]. By random SF networks, we mean the SF network with no degree-degree correlation. The rate equation is set up for the case that updating of the fitness values is carried out not only at the vertex with the smallest fitness value but also at its nearest neighbors, which is called the quenched case. We also compare the quenched case with

the annealed case, where updating is carried out at the vertex with the minimum fitness values as well as the vertices randomly chosen over the entire system and its number is equal to the degree of the vertex with the minimum fitness value. It is noteworthy that the number of vertices updated is not constant in complex networks, but depends on the degree of the vertex with the minimum fitness value. Thus, the analytic approach of the BS model is not as simple as the case in the Euclidean space. Here by applying the rate equation approach as well as the random walk approach, we obtain the fitness threshold analytically to be $x_c = 1/(\langle k \rangle_f + 1)$, where $\langle k \rangle_f = \langle k^2 \rangle / \langle k \rangle$ in the quenched case and $\langle k \rangle_f = \langle k \rangle$ in the annealed case. The avalanche size distribution is obtained to be $P_a(s) \sim s^{-\tau}$ with $\tau = 3/2$ at x_c for $\gamma > 3$.

II. RATE EQUATION APPROACH

In SF networks, it would be essential to take into account the fact that vertices with different degrees experience differ-

ent updating frequencies. Let us denote by f_k the probability that the vertex with the smallest fitness value has degree k. $\rho_k(x)$ is the distribution function of fitness values at the vertices with degree k. Here we set up the master equation for the fitness distribution for the quenched updating, following Ref. [4]. First we define a quantity $Q_k(x)$, the accumulative distribution of the fitness values at vertices with degree k,

$$Q_k(x) = \int_x^1 dx' \rho_k(x'). \tag{1}$$

Thus the fitness distribution $\rho_k(x) = -(\partial/\partial x)Q_k(x)$. Then, we have

$$f_k = -\int_0^1 dx' \frac{\partial}{\partial x'} \{Q_k(x')\}^{Np_k} \prod_{k' \neq k} \{Q_{k'}(x')\}^{Np_{k'}}, \tag{2}$$

where p_k denotes the degree distribution $P_d(k)$ for simplicity. Note that $\Sigma_k f_k = 1$ is satisfied. The evolution equation for the fitness distribution at vertices with degree k is written as

$$\rho_k(x, t+1) = \rho_k(x, t) - \frac{f_k}{Np_k} \left(\frac{-\frac{\partial}{\partial x} \{Q_k(x)\}^{Np_k} \prod_{k' \neq k} \{Q_{k'}(x)\}^{Np_{k'}}}{f_k} \right)$$

$$- \sum_{k''} k'' f_{k''} \frac{kp_k}{\langle k \rangle} \left(\frac{\rho_k(x, t) - \frac{f_k}{Np_k} \left(-\frac{\partial}{\partial x} \{Q_k(x)\}^{Np_k} \prod_{k' \neq k} \{Q_{k'}(x)\}^{Np_{k'}} \frac{1}{f_k} \right)}{Np_k - f_k} \right) + \frac{f_k}{Np_k} + \sum_{k''} \frac{kp_k}{\langle k \rangle} \frac{f_{k''} k''}{Np_k}, \tag{3}$$

where the second term on the right-hand side (RHS) of Eq. (3) represents the update of the minimum fitness when it locates at the vertex with degree k. The third term does the update of the fitness value of a vertex with degree k induced by a nearest neighboring vertex with degree k'' which has the minimum fitness value in the system. The factor $kp_k/\langle k \rangle$ comes from the conditional probability $P(k|k'')$ that the ver-

tex with degree k is connected to the one with k'', which is relevant to the quenched case. In the annealed case, the factor simply is replaced with p_k. The last two terms do the addition of new fitness values [4].

The stationary solution in the limit $t \to \infty$ can be solved by using $\rho_k(x) = -(\partial/\partial x)Q_k(x)$ and taking the integral over $[x, 1]$ of the whole formula as the integral equation,

$$-\int_x^1 dx' \frac{\partial}{\partial x'} \{Q_k(x')\}^{Np_k} \prod_{k' \neq k} \{Q_{k'}(x')\}^{Np_{k'}} \left(-\frac{1}{Np_k} + \frac{k\langle k \rangle_f}{\langle k \rangle (Np_k - f_k)N} \right) - \frac{\langle k \rangle_f kp_k}{\langle k \rangle (Np_k - f_k)} Q_k(x) + \frac{f_k + \frac{kp_k \langle k \rangle_f}{\langle k \rangle}}{Np_k} (1-x) = 0, \tag{4}$$

where

$$\langle k \rangle = \sum_{k=1}^{\infty} kp_k \quad \text{and} \quad \langle k \rangle_f = \sum_{k=1}^{\infty} kf_k. \tag{5}$$

As done in Ref. [4], the threshold x_c is determined by the comparison of the first term with the second term of Eq. (4) in their absolute magnitudes. To proceed, let us first assume that the second term is dominant compared with the first term. Then, we obtain that within the leading order,

$$Q_k(x) \approx \left(\frac{\langle k \rangle f_k}{\langle k \rangle_f k p_k} + 1 \right)(1-x), \qquad (6)$$

which leads to

$$\rho_k(x) \cong \frac{\langle k \rangle f_k}{\langle k \rangle_f k p_k} + 1. \qquad (7)$$

This result holds when $Q_k(x)$ is less than 1 by more than $O(1/N)$. In other words,

$$x \gg x_{c,k} + O(1/N) = \frac{\langle k \rangle f_k}{\langle k \rangle f_k + \langle k \rangle_f k p_k} + O(1/N), \qquad (8)$$

where $x_{c,k}$ is the threshold for a given k. Equation (7) indicates that $\rho_k(x)$ does not vanish as $N \to \infty$. We show that $x_{c,k}$ does not depend on k in the Appendix. Thus we denote $x_{c,k} = x_c$ simply.

For $x < x_c$, the second term of Eq. (4) can be ignored and we have

$$\int_x^1 dx' \frac{Np_k \rho_k(x')}{Q_k(x')} \prod_{k'} \{Q_{k'}(x')\}^{Np_{k'}} \left(1 - \frac{k \langle k \rangle_f}{\langle k \rangle N} \right)$$

$$\cong \left(f_k + \frac{k p_k \langle k \rangle_f}{\langle k \rangle} \right)(1-x). \qquad (9)$$

Next we use the fact that the integral $\int_{x_c}^1 \cdots$ is very small and the interval of the integration is replaced by $\int_x^{x_c}$. By setting $x=0$, we have

$$np_k \rho_k \propto f_k. \qquad (10)$$

On the other hand, by differentiating Eq. (9) at $x=0$, we obtain

$$np_k \rho_k = f_k + \frac{k p_k \langle k \rangle_f}{\langle k \rangle}. \qquad (11)$$

Combining Eqs. (10) and (11), we obtain that

$$f_k = \frac{k p_k}{\langle k \rangle}, \qquad (12)$$

which is supported by numerical simulations shown in Fig. 1. Note that the result of Eq. (12) is for the quenched case. In the annealed case, similar calculations lead to $f_k = p_k$. Thus we obtain $\langle k \rangle_f = \langle k \rangle_{f,q} = \langle k^2 \rangle / \langle k \rangle$ in the quenched case and $\langle k \rangle_f = \langle k \rangle_{f,a} = \langle k \rangle$ in the annealed case. $\langle k \rangle_{f,q}$ diverges for $\gamma < 3$ as $\sim N^{(3-\gamma)/(\gamma-1)}$ in the limit $N \to \infty$.

We also have

$$\rho_k = \frac{k(\langle k \rangle_f + 1)}{n \langle k \rangle}. \qquad (13)$$

Thus, $\rho_k \sim O(N^{-(\gamma-2)/(\gamma-1)})$, and converges to 0 as $N \to \infty$. The convergence rate is slower than the rate of $1/N$ that appears in Euclidean space. Finally, from Eq. (8), the threshold can be expressed simply as
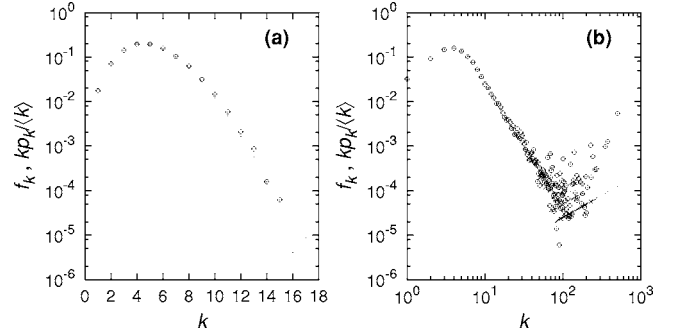

FIG. 1. The data of f_k ($\bigcirc$) and $k p_k / \langle k \rangle$ ($\times$) for the Erdős-Rényi network (a) and for the scale-free network with $\gamma = 3.6$ (b) in the quenched cases. To generate the scale-free network, we use the static model [20]. Both networks have the average degree $\langle k \rangle = 4$ and the system size $N = 10^6$.

$$x_c = \frac{1}{\langle k \rangle_f + 1}, \qquad (14)$$

which does not depend on k for both updating rules. This result is different from the previous results [17,18]. The threshold formula is reproduced by the random walk approach in the next section.

III. RANDOM WALK APPROACH

The random walk approach was first introduced in Refs. [6,7], and it is useful for calculating the avalanche size distribution. The threshold can be also obtained. Let $q_\lambda(t)$ be the probability of having an avalanche with size t, which is defined as the duration of time throughout which the minimum fitness value is smaller than a given threshold value λ. λ can be chosen arbitrarily. Later we find that $q_\lambda(t)$ follows a power law when λ is equal to the threshold x_c. The corresponding generating function is defined as $\chi(z) = \Sigma_{t>0} q_\lambda(t) z^t$. Then $\chi(z)$ satisfies the self-consistent equation [5,7,21],

$$\chi(z) = z g(\chi(z)). \qquad (15)$$

In the previous study [7], the generating function $g(z)$ was given as $\Sigma_t \binom{K}{t} \lambda^t (1-\lambda)^{K-t} z^t = (1-\lambda+\lambda z)^K$ when K fitness values are updated randomly. However, in the case of SF networks, the number of vertices updated at each time step is not constant, but it depends on the degree of the vertex with the minimum fitness value. In this case, the generating function $g(z)$ is given as

$$g(z) = \sum_{k=1}^{\infty} f_{k-1}(1-\lambda+\lambda z)^k, \qquad (16)$$

where f_k was defined as the probability that the minimum fitness locates at the vertex with degree k.

What we do next is to solve $q_\lambda(t)$ by using Eqs. (15) and (16). To proceed, we use Lagrange's inversion formula [22,23],

$$h(\omega) = h(0) + \sum_{n=1}^{\infty} \frac{z^n}{n!} \left(\frac{d^{n-1}}{du^{n-1}} [h'(u)g(u)^n] \right)_{u=0}, \tag{17}$$

where $z = \omega/g(\omega)$, provided that $\omega/g(\omega)$ is analytic near $\omega = 0$ and $h(\omega)$ is an infinitely differentiable function. Here we choose $h(\omega) = \omega$ and $\omega(z) = \chi(z)$. Then,

$$\chi(z) = \sum_{t=1}^{\infty} \frac{z^t}{t!} \frac{\partial^{t-1}}{\partial u^{t-1}} [(g(u))^t]_{u=0}$$

$$= \sum_{t=1}^{\infty} \frac{z^t}{t!} \frac{\partial^{t-1}}{\partial u^{t-1}} \left(\prod_{i=1}^{t} \sum_{k_i=1}^{\infty} f_{k_i-1}(1-\lambda+\lambda u)^{k_i} \right)_{u=0}$$

$$= \sum_{t=1}^{\infty} \frac{z^t}{t!} \frac{\partial^{t-1}}{\partial u^{t-1}} \left[\sum_{k_1=1}^{\infty} \cdots \sum_{k_t=1}^{\infty} \left(\prod_{i=1}^{t} f_{k_i-1} \right)(1-\lambda+\lambda u)^{\Sigma_{i=1}^{t} k_i} \right]_{u=0}$$

$$= \sum_{t=1}^{\infty} \frac{z^t}{t!} \sum_{k_1=1}^{\infty} \cdots \sum_{k_t=1}^{\infty} \left(\prod_{i=1}^{t} f_{k_i-1} \right) \frac{\left(\sum_{i=1}^{t} k_i \right)!}{\left(\sum_{i=1}^{t} k_i - t + 1 \right)!} \lambda^{t-1}(1-\lambda)^{\Sigma_{i=1}^{t} k_i - t + 1}. \tag{18}$$

Since $\chi(z) = \Sigma_t q_\lambda(t) z^t$, $q_\lambda(t)$ can be obtained by using the Stirling's formula as

$$q_\lambda(t) = \frac{1}{t!} \sum_{k_1=1}^{\infty} \cdots \sum_{k_t=1}^{\infty} \left(\prod_{i=1}^{t} f_{k_i-1} \right) \frac{\left(\sum_{i=1}^{t} k_i \right)!}{\left(\sum_{i=1}^{t} k_i - t + 1 \right)!} \lambda^{t-1}(1-\lambda)^{\sum_{i=1}^{t} k_i - t + 1}$$

$$= \sum_{k_1=1}^{\infty} \cdots \sum_{k_t=1}^{\infty} \left(\prod_{i=1}^{t} f_{k_i-1} \right) \frac{\left(\sum_{i=1}^{t} k_i \right)^{\Sigma_{i=1}^{t} k_i}}{\left(\sum_{i=1}^{t} k_i - t \right)^{\sum_{i=1}^{t} k_i - t}} \frac{1}{t' \sum_{i=1}^{t} k_i - t + 1} \frac{\sqrt{2\pi \sum_{i=1}^{t} k_i}}{\sqrt{2\pi \left(\sum_{i=1}^{t} k_i - t \right)} \sqrt{2\pi t}} \lambda^{t-1}(1-\lambda)^{\sum_{i=1}^{t} k_i - t + 1}. \tag{19}$$

Note that k_i is the degree of the vertex with the minimum fitness value at updating time i plus 1, which occurs with the probability f_{k_i-1}. Thus in the limit $t \to \infty$,

$$\sum_{i=1}^{t} k_i \cong t\langle k+1 \rangle_f = (\langle k \rangle_f + 1)t. \tag{20}$$

Substituting Eq. (20) into Eq. (19) yields

$$q_\lambda(t) \cong \frac{1-\lambda}{\lambda} \sqrt{\frac{\langle k \rangle_f + 1}{2\pi \langle k \rangle_f^3}} \left(\frac{\lambda(1-\lambda)^{\langle k \rangle_f}(\langle k \rangle_f + 1)^{\langle k \rangle_f + 1}}{(\langle k \rangle_f)^{\langle k \rangle_f}} \right)^t t^{-3/2}$$

$$+ O(t^{-5/2}). \tag{21}$$

The quantity in the large parentheses is 1 when

$$\lambda = \frac{1}{\langle k \rangle_f + 1}, \tag{22}$$

which is equal to the threshold x_c previously obtained via the rate equation approach. Then, $q_{x_c}(t) \sim t^{-3/2}$. Therefore, the avalanche size distribution behaves as $P_a(s) \sim s^{-3/2}$.

IV. NUMERICAL RESULTS

We check the analytical solution of $q_\lambda(t)$ numerically for several networks. The theoretical formula of x_c is tested through the criticality of the avalanche size distribution. First, the random network with the degree distribution $P_d(k) = \delta_{k,k_0}$, called the regular network, is constructed and the dynamics of the BS model is performed on that network. $k_0 = 4$ is taken for numerical simulations. In this case, x_c reduces to $x_c = 1/(k_0+1)$ simply in both the quenched and annealed cases. In the quenched case [Fig. 2(a)], the avalanche
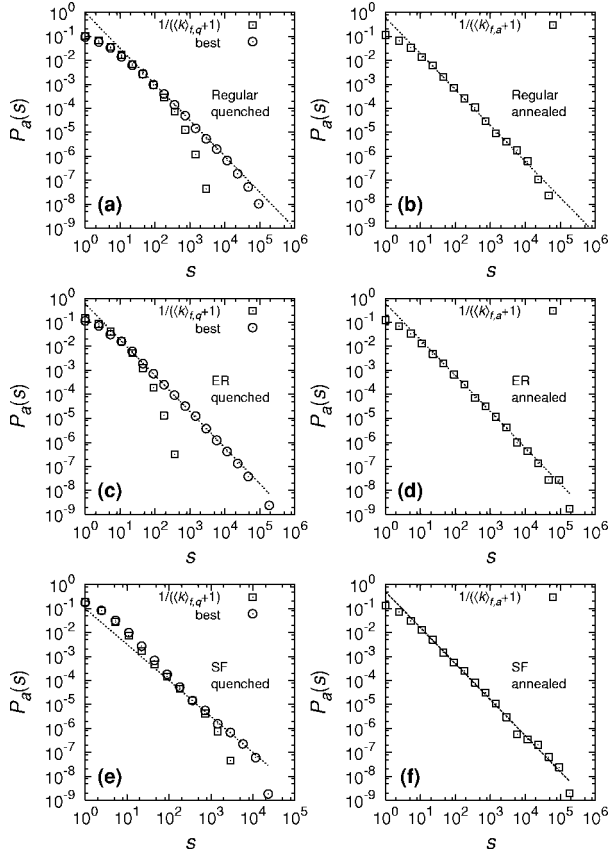
FIG. 2. (a) and (b) The avalanche size distribution for the regular network with the degree distribution of the δ-function. In the quenched case (a), the avalanche size distribution follows a power law when λ is chosen as 0.253 ($\bigcirc$), larger than the theoretical value $1/(\langle k\rangle+1)=0.2$ ($\square$). In the annealed case (b), the theoretical value $x_c=0.2$ ($\bigcirc$) works well to generate the power-law behavior of $P_a(s)$. (c) and (d) Same plot for the Erdős-Rényi network [14]. In the quenched case (c), the power-law behavior of $P_a(s)$ occurs at $x_c=0.207$, larger than the theoretical value, $x_c=1/6$. In the annealed case (d), the theoretical value $x_c=0.2$ generates the power-law behavior. (e) and (f) Same plot for the scale-free network with $\gamma=3.6$. In the quenched case (e), the power-law behavior of $P_a(s)$ occurs at $x_c=0.1$, which is larger than the theoretical value $x_c=0.08$. In the annealed case, the theoretical value $x_c=0.2$ yields the power-law behavior of $P_a(s)$. The mean degree is fixed to be $\langle k\rangle=4$ and the system size is $N=10^6$ in all cases. The straight lines have slope $-3/2$ in all cases.

size distribution does not follow a power law when $\lambda=x_c=0.2$. Instead, the power-law behavior appears at a larger value, $\lambda\approx0.253$. In the annealed case [Fig. 2(b)], however, it follows a power law at $\lambda=x_c$, consistent with the theoretical value.

Second, for Erdős-Rényi (ER) random graph, where the degree distribution is a Poisson distribution, the theoretical formula reduces to $x_c\approx1/(\langle k\rangle+2)$ in the quenched case, because $\langle k^2\rangle=\langle k\rangle^2+\langle k\rangle$ in the limit $N\rightarrow\infty$. Numerical simulations are performed in both the quenched and the annealed cases. In the quenched case [Fig. 2(c)], the avalanche size

distribution does not follow a power law when λ is taken as the theoretical value, but it does when $\lambda\approx0.207$. In the annealed case [Fig. 2(d)], the avalanche size distribution follows a power law at $\lambda=1/(\langle k\rangle+1)$, consistent with the theoretical value.

Next, for SF networks with $\gamma=3.6$, which are constructed by the static model [20], the theoretical value $x_c=1/(\langle k\rangle_f+1)\approx0.08$ in the quenched case. Note that $\langle k\rangle_f\approx11.5$ is different from $\langle k^2\rangle/\langle k\rangle\approx7.06$ numerically due to the strong fluctuations arising in the large k region (Fig. 1). Again the avalanche size distribution does not follow a power law at the theoretical value, but does at $\lambda\approx0.1$. In the annealed case, the avalanche size distribution follows a power law at $x_c=1/(\langle k\rangle_f+1)$.

The numerical results for the above three networks indicate that the mean-field theoretical prediction is not good for the quenched case, however, it is good for the annealed case instead. This result is attributed to the effect of the temporal and spatial correlation between the vertices with the minimum fitness value at successive time steps, which often occur at the nearest neighbors or at the same vertex. Such an effect was not counted properly in the quenched case, and can be neglected in the annealed case.

V. CONCLUSIONS

We have studied the Bak-Sneppen model of complex networks by using the master equation as well as the random walk approaches. The threshold x_c is obtained to be $x_c=1/(\langle k\rangle_f+1)$, where $\langle\cdots\rangle_f$ is the average over the minimum fitness vertices. The avalanche size distribution follows a power law with the exponent $\tau=3/2$ at the critical point. The theoretical prediction of x_c was tested numerically for the regular network, the ER random network, and the SF network. For all the networks, the theoretical predictions of x_c are in disagreement (agreement) with the numerical results for the quenched (annealed) case. The discrepancy in the quenched case is attributed to the effect of the temporal and spatial correlation between the vertices with the minimum fitness values at successive time steps. Nevertheless, the formula $x_c=\langle k\rangle/(\langle k^2\rangle+\langle k\rangle)$ is newly derived here for the quenched case. Thus when $2<\gamma<3$, $x_c\rightarrow0$ in the thermodynamic limit in the quenched case.

APPENDIX

Here we show that $x_{c,k}$ does not depend on k. To proceed, we suppose $x_{k_1,c}<x_{k_2,c}$ for a certain pair of k_1 and k_2. Then, for a given x_0 in the range $x_{k_1,c}<x_0<x_{k_2,c}$, we have with Eq. (4),

$$\int_{x_0}^1 dx' \frac{N p_{k_1} \rho_{k_1}(x')}{Q_{k_1}(x')} \prod_{k'} \{Q_{k'}(x')\}^{N p_{k'}} \left(1 - \frac{k_1 \langle k \rangle_f}{\langle k \rangle N}\right)$$

$$\lesssim \frac{\langle k \rangle_f k_1 p_{k_1}}{\langle k \rangle} Q_{k_1}(x_0), \tag{A1}$$

and

$$\int_{x_0}^1 dx' \frac{N p_{k_2} \rho_{k_2}(x')}{Q_{k_2}(x')} \prod_{k'} \{Q_{k'}(x')\}^{N p_{k'}} \left(1 - \frac{k_2 \langle k \rangle_f}{\langle k \rangle N}\right)$$

$$\cong \frac{\langle k \rangle_f k_2 p_{k_2}}{\langle k \rangle} Q_{k_2}(x_0). \tag{A2}$$

Since $Q_{k_1}(x)$ and $Q_{k_2}(x)$ are of the same order based on Eq. (6), the RHS of Eq. (A1) and that of Eq. (A2) are of the same order. Then we must have $\rho_{k_1}(x') \lesssim \rho_{k_2}(x')$ for $x_0 \le x' \le 1$, which contradicts Eq. (7) for $x > x_{k_2,c} (> x_0, x_{k_1,c})$. Thus we set $x_c = x_{k,c}$ for all k.

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